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Centrifugally driven thermal convection in a rotating cylinder

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Thermally induced convection in a rotating cylinder of fluid heated from above and strongly influenced by centrifugal accelerations is treated using boundary-layer methods. As in the theory of homogeneous rotating fluids, the horizontal Ekman layers control the inviscid axial flow. The solution also largely depends upon the thermal conditions assumed at the side wall, and if these be insulated, consideration of the side-wall boundary layers is necessary for complete specification of the problem. For perfectly conducting side walls, the side layers do not influence the zeroth-order flow, but contribute a second-order correction, which would be absent if the lateral boundaries were ignored. The critical parameters governing the solutions in both cases are found to be γ and the group $\sigma\beta\epsilon^{-\frac{1}{2}}$, where γ is the aspect ratio, σ the Prandtl number, ϵ the Ekman number, and β the thermal Rossby number for the flow. Boundary-layer solutions are given for a wide range of parameters, and gravity is seen to have at most only a local effect on the flow near the side walls.

1. Introduction

It is well known that a density gradient perpendicular to an acceleration will produce motion no matter how small the gradient may be, since the pressure cannot balance the variations in the body force so produced. The most common examples occur, of course, in fluids in a gravitational field which are heated differentially in the horizontal. If the system to be studied undergoes rotation, however, the centrifugal acceleration may play a role analogous to that of gravity in producing motion.

A common example occurs in the case of rotating machinery, where centrifugal accelerations may become quite high. The use of hollow turbine blades to increase heat transfer via centrifugally driven convection has been described by Schmidt (1951). An early analysis of such a thermosyphon was given by Lighthill (1953); in it the centrifugal acceleration was considered constant and the Coriolis acceleration due to rotation was neglected. Such neglect is valid in tubes, where the magnitude of the Coriolis acceleration is limited by geometry; but there exist many cases of fluid motion in which the centrifugal acceleration is spacially varying and the Coriolis acceleration non-negligible. The simplest of these is a heated rotating cylinder of fluid; but only a few pertinent theoretical analyses have appeared in the literature to date.

Centrifugally driven convection subject to variable centrifugal and Coriolis accelerations was considered by Ostrach & Braun (1958). A right circular cylinder of fluid was allowed to rotate about its axis, and was heated differentially in the vertical, i.e. perpendicular to the centrifugal acceleration. The following conclusions were reached: (i) the velocities are small, of order $\alpha\Delta T$, where α is the coefficient of thermal expansion and ΔT is the imposed temperature difference, (ii) the Coriolis force acts to oppose any centrifugally produced radial velocity, and (iii) rotation has little effect on the heat transferred across the cylinder. This last conclusion was deduced from (ii) and a consideration of the energy equation for the system. As we shall see, it is invalid for a boundary-layer flow, and the heat transfer may be augmented by rotation.

Three centrifugally driven problems have recently been considered by Riley (1967). In the first of these, that most important to our discussion, the temperature of a single disk, rotating with its infinite environment is raised (lowered) to a new temperature. There results radial inflow (outflow) in a viscous layer of thickness $O((\nu/\omega)^{\frac{1}{2}})$, where ν is the fluid kinematic viscosity and ω the angular velocity of the disk. The equations yield to a von Kármán (1921) similarity transformation, thus removing the radial dependence, and the initial motion was calculated using a Thiriot (1940) time dependence. In the case of a cooled disk, the induced axial flow is toward the disk and a steady state is possible. For this steady state, a perturbation solution for small changes in wall temperature yields a thin viscous layer imbedded within a 'thermal layer' in which conduction balances the convective effects of the axial flow. For larger temperature differences, the steady state is calculated using the Pohlhausen integral method. Thermally induced 'spin-up' and oscillatory motions were also considered.

Numerical solutions for a cooled disk rotating with its environment are given by Hudson (1968*a*) for high Prandtl number; in this case the thermal and viscous layer thicknesses become of the same order.

Convection in a fluid heated from above and contained between two infinite isothermal coaxial disks has been considered by Hudson (1968*b*). The flow is radially inward in the top Ekman layer and outward in the bottom layer. Linking of the resulting axial flow in the inviscid core is accomplished by Ekman suction due to the thermally induced tangential velocity in the core. The solution was found to depend upon the critical parameter $\sigma\beta\epsilon^{-\frac{1}{2}}$, where σ is the Prandtl number, ϵ is the inverse rotational Reynolds number, and β is the product $\alpha\Delta T$. An asymptotic solution for $\sigma\beta\epsilon^{-\frac{1}{2}} \gg 1$, $\epsilon \ll 0$ was constructed. Thermal layers of dimensionless thickness $O(\epsilon^{\frac{1}{2}}/\sigma\beta)$ form near each disk. The heat transferred to the bottom disk is great, while that transferred from the top disk is zero.

A similar but radially bounded problem was considered by Barcilon & Pedlosky (1967*b*). The inclusion of insulated side walls is seen to have a definite influence upon the flow. Because of the stratification, the side viscous boundary layers no longer have the Stewartson double structure occurring in homogeneous rotating fluids, but a new layer replaces the familiar $\epsilon^{\frac{1}{4}}$ layer because of the domination of gravitational buoyancy over the Coriolis force near the sides. More importantly, consideration of boundary-layer components of the temperature in these layers is necessary for the determination of the flow. Ekman suction, an essential feature of the similarity solutions described above, is absent in some cases. Because of these results, the authors note that unbounded analyses clearly may not represent any physically realizable situation. A perturbation solution for small centrifugal acceleration relative to gravity was given.

Since in many engineering applications centrifugal accelerations may be quite high, we will here treat convection in a cylinder in the limit of an infinitely high ratio of the centrifugal acceleration to that of gravity. Solutions will be given for a large range of parameters, and the conditions under which gravity has no effect on the flow will be deduced.

2. Basic equations

Consider a cylinder of radius a and height $2h$ rotating about its vertical axis with a constant angular velocity ω. The cylinder is assumed to contain a homogeneous Newtonian fluid which is heated isothermally from above. The sides are considered to be either insulated or perfectly conducting, and the rotational rate is assumed to be high, so that the characteristic centrifugal acceleration $\omega^2 a$ is large compared to gravity. The equation of state of the fluid is taken as

$$\rho = \rho_0(1-\alpha(T-T_0)), \tag{2.1}$$

where the subscript denotes a reference quantity, and α is the coefficient of thermal expansion, which is taken as zero in the equations of motion, except when multiplied by either the centrifugal force or gravity. This is a generalization of the Boussinesq approximation to rapidly rotating fluids, and implies that the fluid is incompressible, i.e.

$$\nabla . \mathbf{q} = 0, \tag{2.2}$$

where $\mathbf{q}$ is the velocity. Assuming other fluid properties to be constant, the equations of motion relative to a co-ordinate system rotating with the cylinder become

$$\mathbf{q}.\nabla\mathbf{q} + 2\omega(\mathbf{k}\times\mathbf{q}) + \omega^2 r\alpha(T-T_0)\mathbf{i} - g\alpha(T-T_0)\mathbf{k} = -\rho_0^{-1}\nabla p' + \nu\nabla^2\mathbf{q}, \tag{2.3}$$

where $\mathbf{k}$ and $\mathbf{i}$ are the unit vectors in the vertical and radial directions respectively, g is the acceleration of gravity, ν the fluid kinematic viscosity, and

$$p' = p + z\rho_0 g - \rho_0[\tfrac{1}{2}(\omega^2 r^2)], \tag{2.4}$$

where p' is the perturbation pressure. Defining a polar cylindrical co-ordinate system (r, z, θ) with the origin at the centre of the cylinder, and with corresponding velocity components (u, w, v), and taking the flow to be axially symmetric, a stream function may be defined such that

$$w = -r^{-1}\psi_r, \quad u = r^{-1}\psi_z. \tag{2.5}$$

If T_a and T_b denote the top and bottom temperatures respectively, we choose the following dimensionless quantities (starred),

$$\left.\begin{aligned} & r^* = r/a, \quad z^* = z/h, \\ & T^* = \frac{T-T_0}{\Delta T}, \quad \Delta T = (T_a - T_b)/2, \quad T_0 = (T_a+T_b)/2, \\ & v^* = v\left(\frac{2}{\alpha\Delta T\omega a}\right), \quad \psi^* = \psi\left(\frac{2}{\alpha\Delta T\omega a^2 h}\right). \end{aligned}\right\} \tag{2.6}$$

Cross-differentiating the r and z components of (2.3) to eliminate p', and dropping the stars, the dimensionless equations of motion become

$$\beta\left[r^{-1}\psi_z((r^{-1}\mathscr{L}_\gamma^2\psi)_r - r^{-2}\mathscr{L}_\gamma^2\psi) - r^{-2}\psi_r(\mathscr{L}_\gamma^2\psi)_z - \frac{2vv_z}{r}\right]$$
$$- v_z + rT_z + A\gamma^{-1}T_r = \epsilon r^{-1}\mathscr{L}_\gamma^4\psi, \tag{2.7}$$

$$\beta\left[r^{-2}\frac{\partial(\psi, rv)}{\partial(z,r)}\right] + r^{-1}\psi_z = \epsilon r^{-1}\mathscr{L}_\gamma^2(rv). \tag{2.8}$$

Note that the velocity scale is such that there is an inviscid balance between the Coriolis acceleration and the centrifugal acceleration which causes the motion. The constant property non-dissipative energy equation is

$$\sigma\beta\left[r^{-1}\frac{\partial(\psi, T)}{\partial(z,r)}\right] = \epsilon\nabla_\gamma^2 T. \tag{2.9}$$

In these equations, $\mathscr{L}_\gamma^2$ and ∇_γ^2 are the two-dimensional operators

$$\mathscr{L}_\gamma^2 = \gamma^{-2}r\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \quad \nabla_\gamma^2 = \gamma^{-2}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{2.10}$$

The appropriate boundary conditions are

$$T = \pm 1, \quad z = \pm 1,$$

and either $$T_r = 0, \quad \text{or} \quad T = z, \quad r = 1,$$

together with the dynamical conditions

$$v = \psi = \partial\psi/\partial n = 0$$

on all solid boundaries.

The five important dimensionless parameters entering into the problem are

$\epsilon = \nu/2\omega h^2$ (the Ekman number),

$\sigma = c_p\mu/k$ (the Prandtl number),

$\beta = \frac{1}{4}\alpha\Delta T$ (the thermal Rossby number),

$\gamma = a/h$ (the aspect ratio),

$A = g/\omega^2 a$ (the acceleration ratio).

Many interesting situations fall into the range $\{\epsilon, A, \beta\} \ll 1$, $\{\gamma, \sigma\} \geqslant O(1)$, and an asymptotic solution to the problem will be found for ϵ, $A \to 0$, $\beta \ll 1$. As we shall later deduce, the range of σ for which the solutions are valid will be determined by a restriction on the value of the group $\sigma\beta\epsilon^{-\frac{1}{2}}$.

3. The perturbation expansion

We begin by assuming that the temperature field is given by the conduction solution $T = z$, and solve for the resulting flow field. This method of attack has the advantage of mathematically uncoupling the dynamical variables from the temperature, while elucidating the basic nature of the flow. Then, by calculating a perturbation correction to the conduction profile, important conclusions can be drawn concerning the details of the solution in the general case. In particular, we will deduce that (i) as in the case of homogeneous rotating fluids, the horizontal Ekman layers control the axial flow in the inviscid core, (ii) the boundary layers along the vertical walls of the container have considerable importance in determining the temperature field, and (iii) for sufficiently small β, the solution depends upon β, σ and ϵ only in the product $\sigma\beta\epsilon^{-\frac{1}{2}}$.

Assuming then that $T = z$, the resulting flow equations for $\beta \ll 1$ are

$$-v_z + r = \epsilon r^{-1}\mathscr{L}_\gamma^4 \psi, \tag{3.1}$$

$$\psi_z = \epsilon \mathscr{L}_\gamma^2 (rv). \tag{3.2}$$

Solutions to (3.1) and (3.2) away from the side walls are obtainable by assuming the axial velocity to be a function of z alone, and v to depend linearly upon r. These solutions, which yield the characteristic Ekman layers of thickness $O(\epsilon^{\frac{1}{2}})$, are

$$\left.\begin{aligned} v_0 &= r[z - e^{-\zeta}\cos\zeta + e^{\zeta^*}\cos\zeta^*],\\ \psi_0 &= \frac{\epsilon^{\frac{1}{2}} r^2}{2^{\frac{1}{2}}}[1 - e^{-\zeta}(\cos\zeta + \sin\zeta) - e^{\zeta^*}(\cos\zeta^*(\cos\zeta^* - \sin\zeta^*)],\end{aligned}\right\} \tag{3.3}$$

where $\zeta = (z+1)/(2\epsilon)^{\frac{1}{2}}$ and $\zeta^* = (z-1)/(2\epsilon)^{\frac{1}{2}}$ are the appropriate stretched coordinates near $z = -1$ and $z = +1$ respectively. Note that in balancing the inviscid 'thermal wind', $v = rz$, the well-known Ekman suction induces a weak $O(\epsilon^{\frac{1}{2}})$ downward flow in the core. The radial flow is inward in the upper Ekman layer, outward in the bottom layer, and zero in the core due to Coriolis domination there. These solutions clearly do not apply near the side walls, and must be amended by a further boundary-layer analysis. Layers parallel to the rotation vector were first discussed by Stewartson (1957) and in general have a double structure of thicknesses $\epsilon^{\frac{1}{3}}$ and $\epsilon^{\frac{1}{4}}$. In this case, v is an antisymmetric function of z and only the $\epsilon^{\frac{1}{3}}$ layer arises. Denoting the core plus Ekman layer solutions (3.3) by a subscript 0, we then introduce boundary-layer functions, $\hat{v}$ and $\hat{\psi}$, such that

$$v = v_0 + \hat{v}, \quad \psi = \psi_0 + \hat{\psi}.$$

$\hat{v}$ and $\hat{\psi}$ are then assumed 'steep' near the side walls, and must necessarily decay rapidly away from the boundary. To solve for these boundary-layer corrections to (3.3), the region near the side walls is then divided into the subregions

$$z \pm 1 = O(1), \quad r - 1 = O(\epsilon^{\frac{1}{3}}),$$

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Representations for $\hat{v}$ and $\hat{\psi}$ are constructed for the first two of these subregions, and the requirement that these match asymptotically over adjacent boundaries determines the solutions. The success of this method, first ascribed to Greenspan & Howard (1963) and elaborated upon in detail by Hunter (1967), depends upon the fact that the side layers are thicker than the Ekman layers, and matching is accomplished by consideration of the 'Ekman extensions', $z \pm 1 = O(\epsilon^{\frac{1}{2}})$, $r - 1 = O(\epsilon^{\frac{1}{3}})$. In this manner, consideration of the square region $r - 1 = O(\epsilon^{\frac{1}{2}})$, $z \pm 1 = O(\epsilon^{\frac{1}{2}})$ is avoided. The solutions in the side layer and extensions can then be constructed as a power series in $\epsilon^{\frac{1}{3}}$, and the leading terms, valid away from the top and bottom are

$$\left.\begin{aligned} \hat{v} &= \sum_{n=1}^{\infty} \frac{f_n^{\mathrm{IV}}(\rho)\cos\lambda_n^3(z+1)}{\lambda_n^3} + O(\epsilon^{\frac{1}{3}}), \\ \hat{\psi} &= \epsilon^{\frac{1}{3}} \sum_{n=1}^{\infty} f_n(\rho)\sin\lambda_n^3(z+1) + O(\epsilon^{\frac{1}{2}}), \end{aligned}\right\} \tag{3.4}$$

where

$$\left.\begin{aligned} f_n(\rho) &= \lambda_n^{-7}\left\{\exp(-\lambda_n\rho) - \frac{2}{3^{\frac{1}{2}}}\exp(-\lambda_n\rho/2)\left[\cos\frac{\lambda_n\rho 3^{\frac{1}{2}}}{2} + \frac{\pi}{6}\right]\right\}, \\ \lambda_n^3 &= (2n+1)\pi/2, \\ \rho &= (1-r)\gamma/\epsilon^{\frac{1}{3}}. \end{aligned}\right\} \tag{3.5}$$

The solutions for the extensions are of the Ekman type, i.e.

$$\hat{\psi} = \frac{\epsilon^{\frac{1}{2}}B(\rho)}{2^{\frac{1}{2}}}[1 - e^{-\zeta}(\cos\zeta + \sin\zeta)],$$

$$\hat{v} = B(\rho)[-1 + e^{-\zeta}\cos\zeta],$$

near $z = -1$ and similar expressions near $z = +1$. Matching with the side layers determines the function $B(\rho)$ as

$$B(\rho) = -\sum_{n=1}^{\infty} f_n^{\mathrm{IV}}(\rho)\lambda_n^{-3} + O(\epsilon^{\frac{1}{3}}).$$

It is seen that the Stewartson $\epsilon^{\frac{1}{3}}$ layer must serve a dual purpose. First, it requires an $O(1)$ term in $\hat{v}$ to balance the thermal wind at $r = 1$. Secondly, it must provide a rechanneling of fluid from the bottom Ekman layer to the top one. The magnitude of the required flux is $O(\epsilon^{\frac{1}{2}})$. Since the requirement $\hat{v} = O(1)$ forces $\hat{\psi}$ to be at least $O(\epsilon^{\frac{1}{3}})$, this leading term in $\hat{\psi}$ represents a closed circulation within the Stewartson $\epsilon^{\frac{1}{3}}$ layer. This situation is similar to that observed by Hunter (1967) and appears to be characteristic of many low Rossby number flows which are controlled by Ekman suction and have a non-trivial core angular velocity. In many problems this closed circulation may play only a passive role, but as we shall see, it is of importance in the present problem.

In order to calculate a perturbation correction to the conduction profile, it is necessary to include the effects of convection in (2.9). Thus we set $T = z + \sigma\beta T_{(1)}$, and obtain for $T_{(1)}$

$$\nabla_\gamma^2 T_{(1)} = -(\epsilon r)^{-1}\psi_{(0)r}, \tag{3.6}$$

$$T_{(1)} = 0, \quad z = \pm 1,$$

and either $\partial T_{(1)}/\partial r = 0$, or $T_{(1)} = 0$, $r = 1$, where by $\psi_{(0)}$, we mean the solution $\psi_{(0)} = \psi_0 + \hat{\psi}$ derived above.

Since $\psi_{(0)}$ has components of the boundary-layer type, it follows that $T_{(1)}$ will also have boundary-layer components. The magnitude of $T_{(1)}$ is found by evaluating the source term in each region, which is $O(\epsilon^{\frac{1}{2}})$ in the core and Ekman layers, and $O(\epsilon^{-1})$ in the Stewartson layer. Thus the boundary-layer behaviour of $\psi_{(0)}$ produces components of $T_{(1)}$ which are $O(\epsilon^{-\frac{1}{3}})$ in the Stewartson layer, $O(\epsilon^{\frac{1}{2}})$ in the Ekman layers, as well as a component of $O(\epsilon^{-\frac{1}{2}})$ with $O(1)$ variation due to the core source term. For insulated walls, one would at first expect a contribution of $O(\epsilon^{-\frac{2}{3}})$ from the Stewartson layer requiring a corresponding over-all contribution of this order, but this leading term in the side layer contributes no flux at the wall. As we shall see, this is due to the closed nature of the circulation in this layer to the lowest order. If θ denotes the over-all component, it is obvious that it must satisfy $\theta = 0$ at $z = \pm 1$, since it swamps the Ekman contribution to $T_{(1)}$. Similary, if τ denotes the boundary-layer component of $T_{(1)}$ in the side layer, it is seen to satisfy

$$\tau_{\rho\rho} = \epsilon^{-\frac{2}{3}}\gamma\hat{\psi}_{\rho}, \tag{3.7}$$

which integrates simply to

$$\tau(\rho, z) = \epsilon^{-\frac{2}{3}}\gamma \int_{\infty}^{\rho} \hat{\psi}(\rho, z)\, d\rho. \tag{3.8}$$

In deriving the boundary conditions for θ at $r = 1$, we must include the contributions from τ. If the side walls are conducting, this condition is $\theta(1, z) = 0$, since θ is $O(\epsilon^{-\frac{1}{3}})$ larger than τ. If the walls are insulated, however, the flux due to τ is of the same magnitude as θ, for

$$\tau_r\big|_{r=1} = -\gamma\epsilon^{-\frac{1}{3}}\tau_{\rho}\big|_{\rho=0} = -\frac{\gamma^2}{\epsilon}\hat{\psi}(0, z) = \frac{\gamma^2}{\epsilon^{\frac{1}{2}}2^{\frac{1}{2}}}, \tag{3.9}$$

since $\hat{\psi}(0, z) = -\epsilon^{\frac{1}{2}}/2^{\frac{1}{2}}$ from matching with the core. Thus for insulating walls, we have

$$\left.\begin{aligned} \nabla_{\gamma}^2\theta &= -2^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}, \\ \theta &= 0, \quad z = \pm 1, \\ \theta_r &= -\epsilon^{-\frac{1}{2}}\gamma^2/2^{\frac{1}{2}} \quad (r = 1). \end{aligned}\right\} \tag{3.10}$$

Inclusion of the boundary-layer components of $T_{(1)}$ is therefore necessary to set the problem for the over-all component θ, especially if the side walls are insulated. We will now treat the case of insulated walls in some detail.

The boundary condition at $r = 1$ reflects the fact that convection in the side layer due to the re-channeling of fluid is as important as convection in the core. As we shall see later, the closed circulation also influences the side-layer convection, but is absent to this order in the perturbation. Since this re-channelling must always be present in a bounded system, the effect of side walls is quite striking and cannot be ignored.

The solution for θ is routine, and yields

$$\theta = \epsilon^{-\frac{1}{2}}\left[\frac{1-z^2}{2^{\frac{1}{2}}} - 2^{\frac{1}{2}}\gamma \sum_{n=1}^{\infty} \frac{I_0(\lambda_n^3\gamma r)\sin\lambda_n^3(z+1)}{I_1(\lambda_n^3\gamma)\lambda_n^6}\right]. \tag{3.11}$$

The first term is recognized as the contribution to θ which would arise if the cylinder were unbounded radially, and the second reflects the effect of side walls. It is of interest to see what effect θ has upon the Nusselt number, defined here as

$$Nu(\pm 1) = 2\int_0^1 \frac{\partial T}{\partial z}\bigg|_{z=\pm 1} r\, dr, \tag{3.12}$$

and equal to one for pure conduction. Thus

$$Nu(\pm 1) = 1 \mp \sigma\beta\epsilon^{-\frac{1}{2}}\left[1 - 2\sum_{n=0}^{\infty}\lambda_n^{-6}\right] = 1,$$

a result independent of γ. The corresponding result for infinite coaxial disks which admits a similarity solution, in which T is independent of r, yields

$$Nu(\pm 1) = 1 \mp 2^{\frac{1}{2}}\sigma\beta\epsilon^{-\frac{1}{2}}.$$

Contrasting these results illustrates the profound effect convection in the side layers can have on an enclosed flow of this sort. The vanishing of this correction to the Nusselt number is physically consistent, for noting that θ is an even function of z, the flux across any plane is therefore odd in z. Insulated side walls require that the average flux to the top and bottom plates be equal, which in turn requires the vanishing of

$$\int_0^1 \frac{\partial\theta}{\partial z}\bigg|_{\pm 1} r\,dr,$$

independently of the aspect ratio, since θ is even in z.

The solutions for conducting walls are easily obtainable, but are not given here. However, as we have seen, the closed circulation induces a non-zero value of τ at the wall which must be balanced by a contribution from the core. This correction is $O(\epsilon^{\frac{1}{6}})$ smaller than with insulated walls; thus we see that the thermal conditions imposed at $r = 1$ are of primary importance in determining the solution.

The perturbation may be carried on in a straightforward manner and, if this is done, the expansion is seen to be a power series in the parameter $\lambda = \sigma\beta\epsilon^{-\frac{1}{2}}$ for all dependent variables. With the Rossby number β appropriately defined, this is the same critical parameter which appears in previous analyses of thermally driven rotating flows (Hunter 1967; Duncan 1966; Hudson 1968*b*), and can be interpreted as the ratio of convection to conduction in the core. Indeed, this parameter is expected to be of importance in any thermally driven, axisymmetric, low Rossby number flow which is governed by Ekman suction. Guided by the results of our perturbation analysis, we will be able to construct a solution for $\lambda = O(1)$. However, there are two important modifications which arise in the higher-order terms.

First, the side-layer force balance producing the Stewartson $\epsilon^{\frac{1}{3}}$ layer, i.e. viscous *vs*. Coriolis, is valid only if the term $A\gamma^{-1}T_r$ can be ignored within this layer. This in turn implies $\lambda\gamma A\epsilon^{-\frac{1}{6}} \ll 1$ due to the boundary-layer behaviour of T. This is the same criterion deduced by Barcilon & Pedlosky (1967*a*) and, if violated, requires the appearance of a 'buoyancy layer' replacing the Stewartson $\epsilon^{\frac{1}{3}}$ layer. For the moment we shall assume $\lambda\gamma A\epsilon^{-\frac{1}{6}} \ll 1$, but we will relax this condition in §7.

Secondly, in the case of insulated walls, it becomes necessary for the side layers to have a double structure of thicknesses $\epsilon^{\frac{1}{3}}$ and $\epsilon^{\frac{1}{4}}$. This modification is included in the development of the solution for $\lambda = O(1)$ given in §4.

4. The solution for $\lambda = O(1)$

Using the results of the perturbation analysis, and treating λ as an $O(1)$ parameter, we seek a solution where $T = O(1)$ everywhere with boundary-layer components of $O(\epsilon^{\frac{1}{6}})$, $O(\epsilon^{\frac{1}{4}})$ and $O(\epsilon)$ in the inner and outer side layers and Ekman layers, respectively, $v = O(1)$ everywhere, and $\psi = O(\epsilon^{\frac{1}{2}})$ in the core and Ekman layers, plus the $O(\epsilon^{\frac{1}{3}})$ closed circulation in the inner side layer. Thus with the inertial accelerations formally neglected, we must solve

$$-v_z + r\theta_z + A\gamma^{-1}\theta_r = \epsilon r^{-1}\mathscr{L}_\gamma^4\psi, \tag{4.1}$$

$$\psi_z = \epsilon\mathscr{L}_\gamma^4(rv), \tag{4.2}$$

$$\sigma\beta r^{-1}\frac{\partial(\psi,\theta)}{\partial(z,r)} = \epsilon\nabla_\gamma^2\theta, \tag{4.3}$$

where θ is the largest component of the temperature. With these assumed magnitudes, neglect of the inertial terms is worst in the inner side layers, where the error is $O(\beta\epsilon^{-\frac{1}{3}})$.

In the core, we have the thermal wind relation

$$v = r\theta + A\gamma^{-1}\int_{-1}^{z}\theta_r\,dz + h(r), \tag{4.4}$$

where $h(r)$ is to be determined. The Ekman suction condition, which determines the core axial velocity from the values of the thermal wind at the horizontal boundaries, may be written as

$$\psi\Big|_{z=\pm 1} = \pm\frac{\epsilon^{\frac{1}{2}}}{2^{\frac{1}{2}}}rv\Big|_{z=\pm 1}. \tag{4.5}$$

Insertion of (4.4) into this relation, and using the fact that ψ is independent of z in the core, we obtain for small A,

$$\left.\begin{aligned} v &= r\theta + A\gamma^{-1}\left[\int_{-1}^{z}\theta_r\,dz - \frac{1}{2}\int_{-1}^{1}\theta_r\,dz\right] \doteq r\theta, \\ \psi\epsilon^{-\frac{1}{2}} &= r^2/2^{\frac{1}{2}} + A\gamma^{-1}\left[\frac{1}{2\,2^{\frac{1}{2}}}r\int_{-1}^{1}\theta_r\,dz\right] \doteq r^2/2^{\frac{1}{2}}. \end{aligned}\right\} \tag{4.6}$$

The energy equation in the core becomes

$$-2^{\frac{1}{2}}\lambda\theta_z = \nabla_\gamma^2\theta; \tag{4.7}$$

and, since θ is the largest component of the temperature, it must satisfy the conditions

$$\theta = \pm 1, \quad z = \pm 1.$$

The conditions at $r = 1$ require more discussion, since we have seen that boundary-layer components of the temperature may become equally important. To derive these conditions, we will formally solve the energy equation in the side layers in terms of θ. To do this all variables are expanded in powers of $\epsilon^{\frac{1}{12}}$. Thus for the over-all components,

$$\left.\begin{aligned} \theta(r,z) &= \theta_0 + \epsilon^{\frac{1}{12}}\theta_1 + \dots, \\ v(r,z) &= v_0 + \epsilon^{\frac{1}{12}}v_1 + \dots, \end{aligned}\right\} \tag{4.8}$$

in the inner side layer

$$\left.\begin{aligned} \hat{T}(\rho,z) &= \epsilon^{\frac{1}{6}}\hat{\tau}_1 + \epsilon^{\frac{1}{4}}\hat{\tau}_2 + \epsilon^{\frac{1}{3}}\hat{\tau}_3 + \dots, \\ \hat{\psi}(\rho,z) &= \epsilon^{\frac{1}{3}}\hat{\psi}_0 + \epsilon^{\frac{5}{12}}\hat{\psi}_1 + \dots, \\ \hat{v}(\rho,z) &= \hat{v}_0 + \epsilon^{\frac{1}{12}}\hat{v}_1 + \dots, \end{aligned}\right\} \tag{4.9}$$

where as before $\rho = (1-r)\gamma/\epsilon^{\frac{1}{3}}$,

and in the outer side layer

$$\left.\begin{aligned} \overline{T}(\eta,z) &= \epsilon^{\frac{1}{4}}\overline{\tau}_1+\dots,\\ \overline{v}(\eta,z) &= \overline{v}_0+\dots,\\ \overline{\psi}(\eta,z) &= \epsilon^{\frac{1}{2}}\overline{\psi}_0+\dots,\end{aligned}\right\} \tag{4.10}$$

where $\eta = (1-r)\gamma/\epsilon^{\frac{1}{4}}$.

It is convenient to discuss the velocities in the side layers first, and then proceed to a discussion of the energy equations. In the Stewartson $\epsilon^{\frac{1}{4}}$ layer, the appropriate boundary-layer equations yield

$$\overline{v}_z = 0+O(\epsilon^{\frac{1}{2}}), \quad \overline{\psi}_z = \epsilon^{\frac{1}{2}}\overline{v}_{\eta\eta}. \tag{4.11}$$

Equations (4.11) are not valid within $O(\epsilon^{\frac{1}{2}})$ of the top and bottom, where the solutions in these Ekman extensions are

$$\left.\begin{aligned} \psi &= \frac{\epsilon^{\frac{1}{2}}E(\eta)}{2^{\frac{1}{2}}}[1-e^{\zeta^*}(\cos\zeta^*-\sin\zeta^*],\\ v &= E(\eta)[1-e^{\zeta^*}\cos\zeta^*],\end{aligned}\right\} \tag{4.12}$$

near $z = +1$, and

$$\left.\begin{aligned} \psi &= \frac{\epsilon^{\frac{1}{2}}F(\eta)}{2^{\frac{1}{2}}}[1-e^{-\zeta}(\cos\zeta+\sin\zeta)],\\ v &= F(\eta)[-1+e^{-\zeta}\cos\zeta],\end{aligned}\right\} \tag{4.13}$$

near $z = -1$, where $E(\eta)$ and $F(\eta)$ approach zero as $\eta\to\infty$ in order to match the Ekman layers. Solutions to (4.11) are

$$\overline{v} = C(\eta), \quad \overline{\psi} = \epsilon^{\frac{1}{2}}[C''(\eta)z+D(\eta)],$$

and matching to the Ekman extensions requires

$$E(\eta) = -F(\eta) = C(\eta), \quad D(\eta) = 0, \quad -C''(\eta) = F(\eta)/2^{\frac{1}{2}}, \tag{4.14}$$

relations which may be solved to give

$$\overline{v} = 2^{\frac{1}{2}}\overline{C}\,\exp(-\eta/2^{\frac{1}{4}}), \quad \overline{\psi} = \epsilon^{\frac{1}{2}}\overline{C}z\exp(-\eta/2^{\frac{1}{4}}), \tag{4.15}$$

where $\overline{C} = C_0+\epsilon^{\frac{1}{12}}C_1+\dots$ is an $O(1)$ constant of integration.

The equations for the inner Stewartson layer are

$$-\hat{v}_z = \epsilon^{-\frac{1}{3}}\hat{\psi}_{\rho\rho\rho\rho}, \quad \hat{\psi}_z = \epsilon^{\frac{1}{3}}\hat{v}_{\rho\rho}, \tag{4.16}$$

or, eliminating $\hat{v}$,we have $\hat{\psi}_{zz}+\hat{\psi}_{\rho\rho\rho\rho\rho\rho} = 0$, (4.17)

with boundary conditions at $\rho = \eta = 0$,

$$\left.\begin{aligned} \overline{v}_0+\hat{v}_0 &= -\theta_0(1,z), & \hat{\psi} &= \hat{\psi}_{0\rho} = 0,\\ \overline{v}_1+\hat{v}_1 &= -\theta_1(1,z), & \hat{\psi}_1 &= \hat{\psi}_{1\rho} = 0,\end{aligned}\right\} \tag{4.18}$$

with appropriate matching conditions at $z = \pm 1$. The closed circulation

$$\hat{\psi}_0+\epsilon^{\frac{1}{12}}\hat{\psi}_1$$

vanishes at $z = \pm 1$, so we expand

$$\hat{\psi}_0 + \epsilon^{\frac{1}{12}}\hat{\psi}_1 = \sum_{n=1}^{\infty} [g_{0,n}(\rho) + \epsilon^{\frac{1}{12}} g_{1,n}(\rho)] \sin\frac{n\pi}{2}(z+1). \tag{4.19}$$

The functions, $g_{0,n}(\rho)$ and $g_{1,n}(\rho)$, satisfy

$$g_{i,n}^{\mathrm{VI}} = \omega_n^6 g_{i,n} \quad (i = 0, 1), \qquad \omega_n^3 = \tfrac{1}{2}n\pi, \tag{4.20}$$

from (4.17). The boundary conditions for (4.20) are easily derivable, since

$$\hat{v}_0 + \epsilon^{\frac{1}{12}}\hat{v}_1 = \sum_{n=1}^{\infty} \left[\frac{g_{0,n}^{\mathrm{IV}}(\rho) + \epsilon^{\frac{1}{12}} g_{1,n}^{\mathrm{IV}}(\rho)}{\frac{1}{2}n\pi}\right] \cos\frac{n\pi}{2}(z+1). \tag{4.21}$$

Combining this with (4.15) and (4.18), we get

$$\left.\begin{aligned} g_{i,n}(0) &= g'_{i,n}(0) = 0, \\ g_{i,n}^{\mathrm{IV}}(0) &= -\frac{n\pi}{2}\int_{-1}^{1} \theta_i(1,z)\cos\frac{n\pi}{2}(z+1)\,dz, \\ C_i &= -\frac{1}{2\,2^{\frac{1}{2}}}\int_{-1}^{1} \theta_i(1,z)\,dz, \end{aligned}\right\} \tag{4.22}$$

for $i = 0, 1$. The integrals for $g_{i,n}(\rho)$ are

$$g_{i,n}(\rho) = \frac{g_{i,n}^{\mathrm{IV}}(0)}{2\omega_n^4}\left\{\exp(-\omega_n\rho) - \frac{2}{3^{\frac{1}{2}}}\exp(\tfrac{1}{2}(\omega_n\rho))\cos\left[\frac{3^{\frac{1}{2}}}{2}\omega_n\rho + \tfrac{1}{6}\pi\right]\right\}. \tag{4.23}$$

Up to this point we have determined the leading terms for the velocities in terms of the unknown function θ. This function satisfies the energy equation (4.7), with boundary conditions at $r = 1$ to be derived from a consideration of the energy equations in the side layers. For the outer layer we have

$$\bar{\tau}_{1\eta\eta} = -\frac{\lambda\gamma z C_0}{2^{\frac{1}{4}}}\theta_{0z}(1,z)\exp(-\eta/2^{\frac{1}{4}}), \tag{4.24}$$

or simply

$$\bar{\tau}_1 = -\lambda\gamma z 2^{\frac{1}{4}} C_0 \theta_{0z}(1,z)\exp(-\eta/2^{\frac{1}{4}}). \tag{4.25}$$

Similarily for the inner layers, the expansions (4.8)–(4.9) give

$$\left.\begin{aligned} \hat{\tau}_{1\rho\rho} &= \lambda\gamma\hat{\psi}_{0\rho}\theta_{0z}(1,z), \\ \hat{\tau}_{2\rho\rho} &= \lambda\gamma[\hat{\psi}_{1\rho}\theta_{0z}(1,z) + \hat{\psi}_{0\rho}\theta_{1z}(1,z)], \\ \hat{\tau}_{3\rho\rho} &= \lambda\gamma\left\{\sum_{n=0}^{2}\hat{\psi}_{2-n\rho}\theta_{n_z}(1,z) + \frac{\partial(\hat{\psi}_0,\hat{\tau}_1)}{\partial(\rho,z)}\right\}. \end{aligned}\right\} \tag{4.26}$$

Now for conducting walls, θ_0 remains the largest component of the temperature, and satisfies

$$\left.\begin{aligned} -2^{\frac{1}{2}}\lambda\theta_{0_z} &= \nabla_\gamma^2\theta_0, \\ \theta_0 = \pm 1, \quad z = \pm 1; &\qquad \theta_0 = z, \quad r = 1. \end{aligned}\right\} \tag{4.27}$$

Furthermore, we deduce that $\theta_1 \equiv 0$, and θ_2 satisfies

$$\left.\begin{aligned} -2^{\frac{1}{2}}\lambda\theta_{2_z} &= \nabla_\gamma^2\theta_2 \quad (\theta_2 = 0, \quad z \pm 1), \\ \theta_2 &= -\hat{\tau}_1(0,z), \quad r = 1. \end{aligned}\right\} \tag{4.28}$$

In the case of insulated walls, we have

$$\theta_{0_r} = \gamma[\hat{\tau}_{1\eta}|_{\eta=0} + \hat{\tau}_{3\rho}|_{\rho=0}, \tag{4.29}$$

and the problem for θ_0 becomes,

$$\left.\begin{aligned} -2^{\frac{1}{2}}\lambda\theta_{0_z} &= \nabla_\gamma^2\theta_0 \quad (\theta_0 = \pm 1, \quad z = \pm 1), \\ \theta_{0_r} &= \lambda\gamma^2\left\{-\frac{\theta_{0_z}(1,z)}{2^{\frac{1}{2}}} + \int_\infty^0 \frac{\partial(\hat{\psi}_0, \hat{\tau}_1)}{\partial(\rho, z)}d\rho\right\}, \quad (r = 1), \end{aligned}\right\} \tag{4.30}$$

where we have made use of the fact that

$$\hat{\psi}_2(0,z) + C_0 z = -1/2^{\frac{1}{2}}. \tag{4.31}$$

Note that the form of the boundary condition (4.29) restricts the range of γ to $\gamma = O(1)$, whereas in the case of conducting walls it may be shown that the results are valid for $\gamma \leqslant O(\epsilon^{-\frac{1}{3}})$. The effect of the closed circulation $\hat{\psi}_0$, as well as the re-channelling of fluid, is felt directly in the case of insulated walls, for it appears in the boundary conditions for θ_0. For conducting walls, this circulation requires an $O(\epsilon^{\frac{1}{3}})$ correction to θ_0, which would be absent if the side wall layers were not considered. In both cases we see that the inclusion of side layers leads to substantial effects and renders suspect theories which *a priori* neglect lateral boundaries.

5. The solution for conducting walls

The solution for this case is easily obtained, since the leading term θ_0 uncouples from the closed circulation $\hat{\psi}_0$ which it produces. For the solution of (4.27) we set $\theta_0 = z + \Phi(r,z)$ and thus obtain

$$\left.\begin{aligned} -2^{\frac{1}{2}}\lambda(1+\Phi_z) &= \nabla_\gamma^2\Phi, \\ \Phi = 0, \quad z = \pm 1, &\qquad \Phi = 0, \quad r = 1. \end{aligned}\right\} \tag{5.1}$$

If we then assume the Fourier–Bessel representations,

$$\Phi = \sum_{n=1}^{\infty}\Phi_n(z)J_0(\alpha_n r), \quad 1 = \sum_{n=1}^{\infty} b_n J_0(\alpha_n r), \tag{5.2}$$

where $$J_0(\alpha_n) = 0 \quad (n = 1, 2 \ldots),$$

(5.1) yields an ordinary differential equation for the $\Phi_n(z)$, to be solved with the conditions $\Phi_n(\pm 1) = 0$. These solutions are

$$\Phi_n(z) = a_n^+ \exp(r_n^+ z) + a_n^- \exp(r_n^- z) + 2^{\frac{1}{2}}\lambda(\alpha_n/\gamma)^{-2}b_n, \tag{5.3}$$

where
$$b_n = 2/\alpha_n J_1(\alpha_n),$$
$$r_n^\pm = \{-\lambda(2)^{\frac{1}{2}} \pm [2\lambda^2 + 4(\alpha_n/\gamma)^2]^{\frac{1}{2}}\}/2,$$
$$a_n^\pm = \pm 2^{\frac{1}{2}}\lambda b_n(\alpha_n/\gamma)^{-2}\sinh(r_n^\mp)/\sinh(r_n^+ - r_n^-).$$

Evaluation of the leading contribution to the Nusselt number,

$$Nu^{(0)}(\pm 1) = 2\int_0^1 \left.\frac{\partial\theta_0}{\partial z}\right|_{z=\pm 1} r\,dr,$$

gives

$$Nu^{(0)}(\pm 1) = 1 + 4(2)^{\frac{1}{2}}\lambda\gamma^2 \sum_{n=1}^{\infty} \frac{r_n^+ \sinh(r_n^-)\exp(\pm r_n^+) - r_n^- \sinh(r_n^+)\exp(\pm r_n^-)}{\alpha_n^4 \sinh(r_n^+ - r_n^-)}, \tag{5.4}$$

which for $\gamma \to \infty$ yields the explicit sum

$$Nu^{(0)}(\pm 1) = \frac{2^{\frac{1}{2}}\lambda \exp(\mp 2^{\frac{1}{2}}\lambda)}{\sinh 2^{\frac{1}{2}}\lambda}. \tag{5.5}$$

The results of numerical evaluation of the series are shown in figure 1. Note that at even moderate values of λ, the flux from the top plate falls toward zero, while the flux to the bottom plate increases significantly with increasing λ. These results

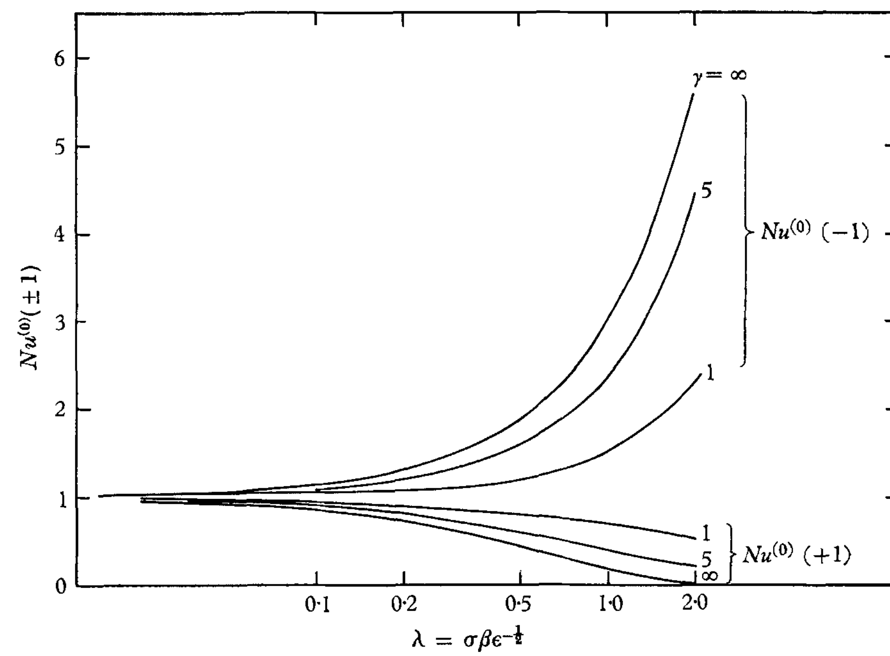

FIGURE 1. Zeroth-order Nusselt numbers for conducting side walls.

correspond to the gradual appearance of a 'thermal layer' near the bottom plate while the core temperature approaches that of the top plate, Hudson (1968*b*). Indeed, for $\gamma \to \infty$, (5.5) gives results identical to the similarity solution for infinite disks for large λ, Hudson (1968*b*), and reduces to the perturbation result

$$Nu^{(0)}(\pm 1) = 1 \mp 2^{\frac{1}{2}}\lambda \tag{5.6}$$

for small λ. As we have seen, these results are physically admissible, since the flux from the top need not balance that to the bottom.

Although the isotherms are swept down in the core, θ_0 is (locally) an odd function of z in the side layers, since the condition $\theta_0(1, z) = z$ is valid for all λ. Hence the solution for the side layers is identical to the flow due to conduction and the $\epsilon^{\frac{1}{4}}$ outer layer does not appear here. The leading terms in the inner layer are given by (3.4), which then sets the problem for θ_2, since the condition

$$\theta_2(1,z) = -\hat{\tau}_1(0,z) = -\lambda\gamma\int_{\infty}^{0} \hat{\psi}_0(\rho,z)\,d\rho \tag{5.7}$$

is now known. Series solutions for θ_2 have been obtained, but are not quoted here for the sake of brevity. However the $\epsilon^{\frac{1}{8}}$ corrections to the Nusselt number are shown in figure 2 and show the same qualitative features as the zeroth-order results.

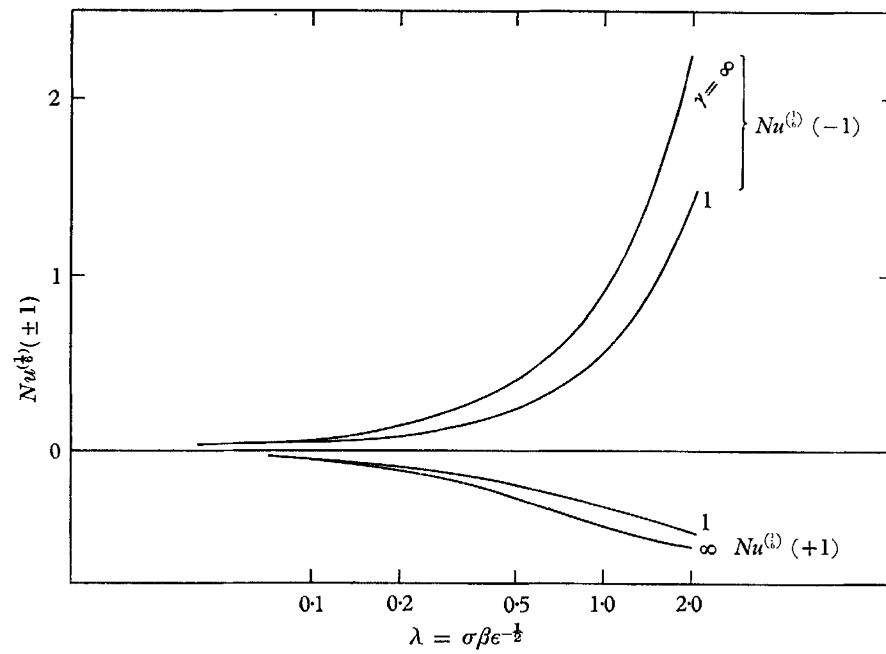

FIGURE 2. $\epsilon^{\frac{1}{8}}$ corrections to the Nusselt numbers for conducting side walls.

6. The solution for insulated walls

From the results of §4, θ_0 satisfies (4.30). Using the fact that

$$\hat{\tau}_1 = \lambda\gamma\int_{\infty}^{\rho} \hat{\psi}_0(\rho,z)\,d\rho\, \theta_{0z}(1,z), \tag{6.1}$$

we may eliminate $\hat{\tau}_1$ from the boundary condition to obtain (after one integration by parts)

$$\theta_{0_r} = -\lambda\gamma^2\left[\frac{\theta_{0_z}}{2^{\frac{1}{2}}} + \lambda\gamma\frac{d}{dz}\left\{\theta_{0_z}\int_{\infty}^{0} \hat{\psi}_0^2(\rho,z)\,d\rho\right\}\right] \quad (r = 1). \tag{6.2}$$

This is a non-linear problem, since $\hat{\psi}_0$ is coupled to θ_0 through the value of the thermal wind at $r = 1$. Indeed, $\hat{\psi}_0$ arises solely from the requirement that $\hat{v} = O(1)$ to balance this thermal wind within the $\epsilon^{\frac{1}{4}}$ layer. Expansion in powers of λ linearizes the problem, but soon runs into trouble due to the complicated integrals appearing in the boundary conditions and the increasing complexity of the convection term in (4.30). Thus any solution will in general be numerical. We set $\theta = z + \Phi(r,z)$ to obtain

$$\left.\begin{aligned} -2^{\frac{1}{2}}\lambda(1+\Phi_z) &= \nabla^2_\gamma \Phi \quad (\Phi = 0, \quad z = \pm 1), \\ \Phi_r &= -\lambda\gamma^2\left[\frac{1+\Phi_z}{2^{\frac{1}{2}}} + \lambda\gamma\frac{d}{dz}\left\{(1+\Phi_z)\int_\infty^0 \hat{\psi}_0^2(\rho,z)\,d\rho\right\}\right] \quad (r = 1). \end{aligned}\right\} \tag{6.3}$$

Consider now the Sturm–Liouville system

$$\chi_p'' + 2^{\frac{1}{2}}\lambda\chi_p' = -\beta_p^2\chi_p \quad (\chi_p(\pm 1) = 0), \tag{6.4}$$

with λ given. The resulting eigenfunctions and eigenvalues are

$$\left.\begin{aligned} \chi_p &= \exp(-\lambda z/2^{\frac{1}{2}})\sin p[\tfrac{1}{2}\pi](z+1), \\ \beta_p^2 &= \lambda^2/2 + (p[\tfrac{1}{2}\pi])^2 \quad (p = 1, 2, \ldots). \end{aligned}\right\} \tag{6.5}$$

By the usual Sturm–Liouville theorems, these eigenfunctions form a complete set on $(-1, 1)$ with the orthogonality property

$$\int_{-1}^{1} \exp(2^{\frac{1}{2}}\lambda z)\,\chi_m(z)\,\chi_n(z)\,dz = \delta_{m,n}, \tag{6.6}$$

where $\delta_{m,n}$ is the Kronecker delta. We now assume the series representations,

$$\Phi = \sum_{n=1}^{\infty} \Phi_n(r)\,\chi_n(z), \quad 1 = \sum_{n=1}^{\infty} a_n\chi_n(z), \tag{6.7}$$

which, when substituted into the first of (6.3), yield a differential equation for Φ_n whose solution gives,

$$\Phi_n(r) = c_n I_0(\beta_n\gamma r) + d_n K_0(\beta_n\gamma r) + \frac{2^{\frac{1}{2}}\lambda a_n}{\beta_n^2}. \tag{6.8}$$

We conclude that $d_n = 0$ since the region is simply connected, and the c_n are then determined by the remaining boundary condition at $r = 1$,

$$\begin{aligned} \sum_n c_n\beta_n I_1(\beta_n\gamma)\chi_n(z) &= -\frac{\lambda\gamma}{2^{\frac{1}{2}}}\left[1 + \sum_n \Phi_n(1)\chi_n'(z)\right] \\ &\quad - (\lambda\gamma)^2\frac{d}{dz}\left[\int_\infty^0 \hat{\psi}^2 d\rho\left(1 + \sum_n \Phi_n(1)\chi_n'(z)\right)\right]. \end{aligned} \tag{6.9}$$

If we then multiply (6.9) by $\exp(2^{\frac{1}{2}}\lambda z)\chi_m(z)$ and integrate from -1 to 1, an infinite set of non-linear algebraic equations for the c_n is generated. This set was programmed for iterative solution on an IBM 7094, truncated and solved. It is to be emphasized that all of the integrals arising on the right-hand side of (6.9) were evaluated analytically and were furnished as input data for the program. Only the c_n were obtained numerically. The iteration was started by assuming $\Phi = \hat{\psi}_0 = 0$ on the right-hand side of (6.9), which gave an initial guess for the c_n. Further estimates were then computed by evaluating the entire right-hand side using the previous values of the c_n. Enough terms were included to ensure the accuracy of the leading coefficients, and in general fewer than 20 iterations were necessary for convergence of the estimates. The results are given in table 1 for $\lambda = 0{\cdot}5$ and $1{\cdot}0$ for an aspect ratio of 1.

Shown in figure 3 are the temperature profiles for these two values of λ. The profiles are swept downward in the core due to unidirectional convection there and are similarily swept upwards near the sides; this reflects the effect of the re-channeling of fluid along these walls.

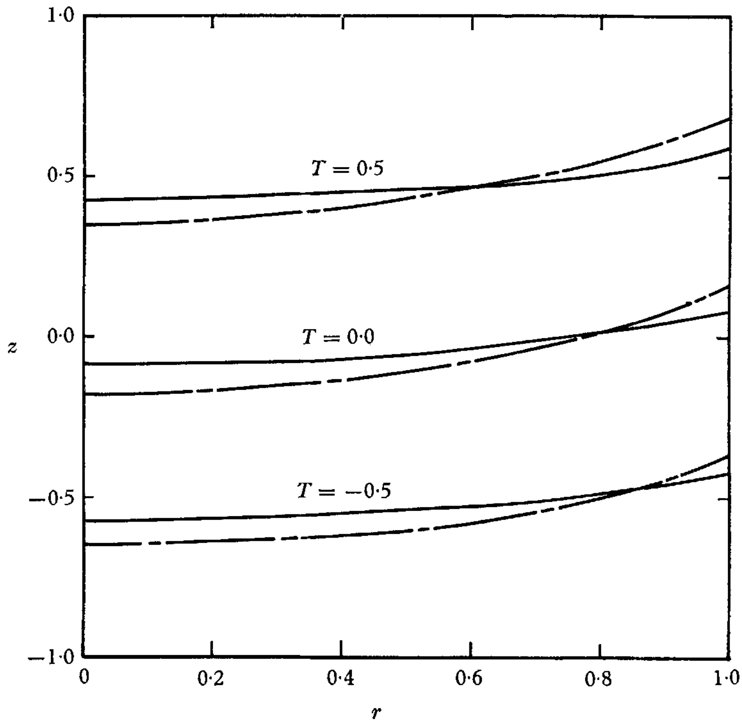

FIGURE 3. Temperature profiles for the case of insulated side walls for $\gamma = 1{\cdot}0$. ——, $\lambda = 0{\cdot}5$; – – –, $\lambda = 1{\cdot}0$.

Although the series for Φ converges satisfactorily, the differentiated series for $\Phi_z(r, \pm 1)$ converges only slowly, so that no values of the Nusselt number for insulated walls are obtainable by this method. An expansion in λ, however, indicates that for small λ,

$$Nu(\pm 1) = 1 + S(\gamma)\lambda^2 + O(\lambda^4),$$

where $S(\gamma)$ is a function of γ alone.

7. The solution for $A \gg \epsilon^{\frac{1}{3}}$

As noted in §3, the restriction $\lambda\gamma A\epsilon^{-\frac{1}{3}} \ll 1$ was necessary to maintain the balance between viscous and Coriolis forces which produces the Stewartson $\epsilon^{\frac{1}{3}}$ layer. If for λ, $\gamma = O(1)$ we reverse the condition, i.e. consider $\epsilon^{\frac{1}{3}} \ll A \ll 1$, the structure of the side layers also changes. This illustrates the fact that when constructing an asymptotic (boundary-layer) solution when two parameters (A and ϵ) approach zero simultaneously, the solution often depends critically upon their ratio. Furthermore, the exact condition can be deduced from the differential equations only after a solution has been constructed.

Including the effects of buoyancy in the side layers, we must treat the approximate set

$$-\hat{v}_z + A\gamma^{-1}\tau_r = \epsilon\hat{\psi}_{rrrr}, \tag{7.1}$$

$$\hat{\psi}_z = \epsilon\hat{v}_{rr}, \tag{7.2}$$

$$\tau_{rr} = -\frac{\sigma\beta}{\epsilon}\hat{\psi}_r\theta_{0z}(1,z), \tag{7.3}$$

where we have assumed $\tau < O(1)$ in (7.3). The Stewartson $\epsilon^{\frac{1}{4}}$ layer will again appear, since it arises solely from the requirements (from the core) that $\bar{v} = O(1)$ and $\bar{\psi} = O(\epsilon^{\frac{1}{2}})$. A balance between buoyancy and viscous terms in (7.1) yields a 'buoyancy layer' of thickness $\epsilon^{\frac{1}{2}}/(\sigma\beta A)^{\frac{1}{4}}$ similar to that found in Barcilon & Pedlosky (1967*a*). The scaling in this layer is

$$\hat{\psi} = O(\epsilon^{\frac{1}{2}}), \quad \hat{v} = O\left(\frac{\epsilon}{\sigma\beta A}\right)^{\frac{1}{2}} \ll O(1).$$

Because the $O(1)$ thermal wind must be balanced by boundary-layer contributions having sufficiently arbitrary z variation, a third balance in (7.1) is necessary between the Coriolis and buoyancy terms. This third balance leads to a 'hydrostatic' layer of thickness $O(\sigma\beta A)^{\frac{1}{2}}$, i.e. an intermediate layer between the $\epsilon^{\frac{1}{4}}$ and buoyancy layers. This triple structure is similar to the situation found in Barcilon & Pedlosky (1967*a*), but, as in the case $A \ll \epsilon^{\frac{1}{6}}$ treated in §4, there arises a closed circulation of $O(\epsilon/\sigma\beta A) > O(\epsilon^{\frac{1}{2}})$ due to the balancing of the $O(1)$ thermal wind within the hydrostatic layer.

In this case, it can be shown that the asymptotic expansion is in powers of $A^{\frac{1}{2}}$ rather than $\epsilon^{\frac{1}{12}}$ in the case of Coriolis domination of the side layers. We further remark that when $A \sim \epsilon^{\frac{1}{6}}$, the hydrostatic and buoyancy layers merge into a single layer of $O(\epsilon^{\frac{1}{3}})$, the closed circulation becomes $O(\epsilon^{\frac{1}{3}})$, and the expansion is again in $\epsilon^{\frac{1}{12}}$; but we only treat the two limiting cases, $A \gtrless \epsilon^{\frac{1}{6}}$. When the formal expansions are substituted into the pertinent boundary-layer equations and like powers of A equated, the formulation is quite similar to that obtained in §4 with one important modification: although there is a closed circulation, it has only a minor effect on the boundary conditions for θ_0. In the case of conducting walls, it induces a correction to the zeroth-order results of §5 (which remain valid) of only $O([\sigma\beta/A]^{\frac{1}{2}}) \ll \epsilon^{\frac{1}{6}}$. In the case of insulated walls, this circulation does not appear in the boundary condition for θ_{0_r}, which is

$$\theta_{0_r} = \frac{-\lambda\gamma^2\theta_{0_z}(1,z)}{2^{\frac{1}{2}}} \quad (r = 1). \tag{7.4}$$

This reflects only the convection due to the re-channelling of fluid, which of course must always be present. Thus for $A \gg \epsilon^{\frac{1}{6}}$, convection in the side layers remains important, but the closed circulation plays a passive role.

Since θ_0 still satisfies the first two of (4.30), we again assume

$$\theta_0 = z + \Phi(r,z) = z + \sum_n \Phi_n(r)\chi_n(z),$$

where

$$\Phi_n(r) = c_n I_0(\beta_n\gamma r) + 2^{\frac{1}{2}}\lambda a_n/\beta_n^2,$$

and then the transformed boundary condition at $r = 1$ yields the infinite set of linear algebraic equations for the c_n,

$$c_n \beta_n I_1(\beta_n \gamma) = \frac{-\lambda\gamma}{2^{\frac{1}{2}}}\left[a_n + \sum_{m=1}^{\infty}\left(c_m I_0(\beta_m\gamma) + \frac{2^{\frac{1}{2}}\lambda a_m}{\beta_m^2}\right) d(m,n)\right], \tag{7.5}$$

where
$$d(m,n) = \int_{-1}^{1} \exp(2^{\frac{1}{2}}\lambda z)\, \chi'_m(z)\, \chi_n(z)\, dz.$$

In this case the numerical solution was much easier; the set was truncated and solved by a simple matrix inversion. The c_n for this case were calculated taking progressively more terms until the leading terms showed no dependence upon the order of the system. In this way an initial estimate of the number of terms

$A \ll \epsilon^{\frac{1}{6}}$, $\lambda = 0{\cdot}5$	$A \ll \epsilon^{\frac{1}{6}}$, $\lambda = 1{\cdot}0$	$A \gg \epsilon^{\frac{1}{6}}$, $\lambda = 1{\cdot}0$
−0·257 (00)	−0·438 (00)	−0·440 (00)
0·795 (−2)	0·277 (−1)	0·262 (−1)
−0·194 (−2)	−0·512 (−2)	−0·436 (−2)
0·141 (−3)	0·600 (−3)	0·511 (−3)
−0·402 (−4)	−0·129 (−3)	−0·997 (−4)
0·376 (−5)	0·177 (−4)	0·142 (−4)
−0·109 (−5)	−0·410 (−5)	−0·288 (−5)
0·117 (−6)	0·606 (−6)	0·448 (−6)
−0·339 (−7)	−0·144 (−6)	−0·924 (−7)
0·397 (−8)	0·204 (−7)	0·152 (−7)
−0·115 (−8)	−0·499 (−8)	−0·316 (−8)
0·146 (−9)	0·847 (−9)	0·540 (−9)
−0·427 (−10)	−0·306 (−9)	−0·113 (−9)
0·587 (−11)	0·702 (−10)	0·198 (−10)
−0·167 (−11)	−0·163 (−10)	−0·413 (−11)

TABLE 1. The c_n for $\gamma = 1{\cdot}0$

necessary for the accurate solution of the more complicated non-linear set of § 6 was obtained. The c_n for $\lambda = 1, \gamma = 1$ are tabulated in table 1, and comparison with the coefficients for $A \gg \epsilon^{\frac{1}{6}}$ shows that the two solutions are nearly identical. Although not shown here, the isotherms for $A < \epsilon^{\frac{1}{6}}$, $\lambda = \gamma = 1{\cdot}0$ exhibit a slightly more pronounced upsweep near the side walls compared to the results in figure 1 for $\lambda = 1{\cdot}0$. The difference is of the order of 3 % or less and is due to the absence of the closed circulation to this order. Hence, although the structure of the viscous side layers is radically different in the two cases $A \lesseqgtr \epsilon^{\frac{1}{6}}$, the main effect of these layers upon the over-all temperature arises from the necessary rechannelling of fluid from bottom to top.

Because of the numerical simplicity of the method used to solve (7.5), many more coefficients than necessary for adequate representation of the temperature could be computed in hopes of obtaining Nusselt numbers for the insulated case, given by

$$Nu^{(0)}(\pm 1) = 1 + \pi \exp(\mp \lambda/2^{\frac{1}{2}}) \sum_{n=1}^{\infty} n(\mp 1)^n \left(\frac{c_n I_1(\beta_n\gamma)}{\beta_n \gamma} + \frac{\lambda a_n}{2^{\frac{1}{2}}\beta_n^2}\right). \tag{7.6}$$

The maximum number of terms which could be handled with the machine storage available was approximately 125, and it was found that the truncated series (7.6) converged well only for values of λ and γ such that the product $\lambda\gamma$ was not much greater than 1·0. Nusselt numbers were computed for $\lambda = 0{\cdot}01$ and $0{\cdot}1$, $1 \leqslant \gamma \leqslant 10$ and the four cases $\lambda = 0{\cdot}5$, $1{\cdot}0$, $\gamma = 1, 2$. The results for small λ were fitted to an expression of the form

$$Nu(\mp 1) = 1 + S(\gamma)\lambda^2 + O(\lambda^4) \tag{7.7}$$

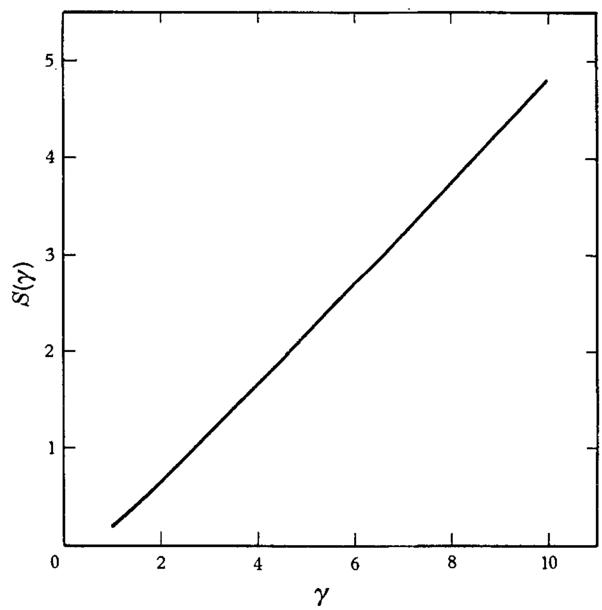

FIGURE 4. The function $S(\gamma)$ for $A > \epsilon^{\frac{1}{8}}$.

suggested by the theory. It may be noted that the series (7.6) evaluated both at $z = \mp 1$ gave identical results as required from physical considerations. $S(\gamma)$, obtained from the numerical results, is shown in figure 4 and is very nearly a linear function of γ. From comparison with results computed for higher values of λ it is believed that neglect of the $O(\lambda^4)$ term in (7.7) yields Nusselt numbers in error by less than 10% for λ only slightly less than 1·0.

8. Conclusions

We have considered thermal convection in a rotating cylinder of fluid heated from above and strongly influenced by the centrifugal acceleration. The horizontal Ekman layers were found to control the axial flow in the inviscid core, and the solution depends upon the thermal conditions at the side walls. For insulated walls, convection due to the re-channelling of fluid from the bottom to the top Ekman layer is as important as convection in the core, and the neglect of side walls leads to spurious results. For conducting walls, the similarity solution properly represents the solution in the double limit, $\epsilon \to 0$, $\gamma \to \infty$, with the effect of side walls appearing as a second-order correction to the zeroth-order results.

Because of the strong influence of the Ekman layers, the zeroth-order solutions for both insulated and conducting side walls are found to depend upon σ, β and ϵ only in the group $\lambda = \sigma\beta\epsilon^{-\frac{1}{2}}$.

For situations such that $\lambda\gamma A \ll \epsilon^{\frac{1}{8}}$ (or in a zero-gravity field) gravitational buoyancy has no effect upon the dynamics, whereas for $A \gg \epsilon^{\frac{1}{8}}$, λ, $\gamma = O(1)$, it has a definite but local effect upon the side layers; that the effect is local was demonstrated *a posteriori*, since the over-all temperature for the cases $A \lesseqgtr \epsilon^{\frac{1}{8}}$ differed only slightly. Solutions have been presented for the range $\epsilon \ll 1$, $A \ll 1$, $\beta \ll \epsilon^{\frac{1}{2}}$, $\lambda \leqslant O(1)$ and

$$\gamma = O(1)\,(\text{insulated walls}), \quad \gamma \leqslant O(\epsilon^{-\frac{1}{8}})\,(\text{conducting walls}),$$

with emphasis on the Nusselt number whenever possible. Actual computation of the heat transfer for conducting walls and for certain cases of insulated walls was possible, and from these results it is concluded that the heat transfer may be considerably augmented by rotation.

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